

Cordial Sets of Hypercubes

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Abstract

For a graph $G = (V, E)$, a binary labeling (coloring) $f : V(G) \rightarrow \mathbb{Z}_2$, is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. The friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)| \forall xy \in E(G)$. Let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges labeled i . The cordial set of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling of } G\}.$$

In this paper we show that all n -cubes Q_n are cordial and determine the cordial sets of hypercubes Q_n ($n \leq 5$). In addition, a conjecture concerning the cordial sets of Q_n ($n > 5$) is presented.

Key Words: Cordial graphs; cordial set; friendly labeling.

AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [7]. Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}_2$ be a binary vertex labeling (or coloring) of G . For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$.

Bulletin of Institute of Combinatorics and its Applications **75** (2015), 95-106.

The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1. Any friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)| \forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges of G that are labeled i . The number $N(f) = |e_f(1) - e_f(0)|$ is called the *cordial index* of f . A graph G is said to be *cordial* if it admits a friendly labeling with cordial index 0 or 1.

To illustrate the above concepts, consider the graph G of Figure 1, which has six vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that three vertices be labeled 0 and the other three 1.

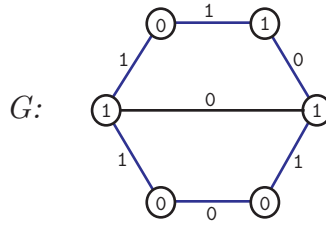


Figure 1: An Example of friendly labeling of G .

Figure 1 also shows the associated edge labeling of G , where four edges have label 1 while the other three edges have labels 0. Therefore, the cordial index of this labeling is $4 - 3 = 1$ and G is cordial. In 1978, I. Cahit [2, 3, 4] introduced the concept of cordial labeling as weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Hovay [9], later generalized the concept of cordial graphs and introduced A -cordial labelings, where A is an abelian group. A graph G is said to be A -cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cairnie-Edwards [5] proved that the problem of deciding whether or not a graph G is cordial is NP-complete, as conjectured by Kirchherr [11]. They specifically proved that for a graph $G = (V, E)$ with even number of edges, the question of whether G admits a cordial labeling is NP-complete. Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 8, 10, 12, 13, 14, 15, 16, 17]. In [6] Chartrand-Lee-Zhang introduced the concept of *friendly index set* (or cordial set) of a graph G . The *cordial set* of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{N(f) : f \text{ is a friendly labeling of } G\}.$$

For the graph G in Figure 1, it is easy to verify that $C(G) = \{1, 3, 7\}$. The friendly colorings of G that provide the other cordial indices are presented in Figure 2.

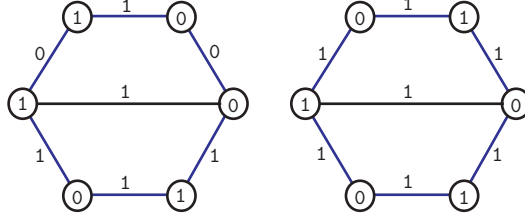


Figure 2: Two friendly labelings of G with indices 3 and 7.

In the next section, we will utilize the concept of cordial set to prove that hypercubes are cordial. Note that if $f : V(G) \rightarrow \mathbb{Z}_2$ is a friendly labeling, so is its inverse labeling $g : V(G) \rightarrow \mathbb{Z}_2$ defined by $g(v) = 1 - f(v) \forall v \in V(G)$. Moreover, $N(g) = N(f)$. A friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ is called *maximum friendly labeling* of G if its cordial index is the maximum element of $C(G)$. In this case, we call $N(f)$ the maximum cordial index of G . For a graph $G = (p, q)$ of order p , size q , and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G , we have

$$N(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \quad (1.1)$$

Consequently, to find the cordial index of f it is enough to find $e_f(1)$ or $e_f(0)$. Moreover, to determine the cordial set of G it is enough to compute $e_f(1)$ [or $e_f(0)$] for different friendly labelings. Another immediate consequence of (1.1) is the following useful fact:

Observation 1.1. For a graph G of size q , $C(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.

2 Hypercubes are Cordial

Hypercubes (or cubes), denoted by Q_n , are defined by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ ($n \geq 2$).

Remark 2.1. Let $\{u_i : 1 \leq i \leq 2^{n-1}\}$ be the vertices of Q_{n-1} and $V(K_2) = \{a, b\}$. Also, let H be the subgraph of Q_n induced by the vertices $\{v_i = (u_i, a) : 1 \leq i \leq 2^{n-1}\}$ and K be the subgraph induced by the vertices $\{w_i = (u_i, b) : 1 \leq i \leq 2^{n-1}\}$. These two subgraphs are isomorphic with Q_{n-1} and Q_n is the disjoint union of H and K with the added edges $e_i = v_i w_i$ ($1 \leq i \leq 2^{n-1}$).

Remark 2.2. The cube Q_n also can be viewed as a graph whose vertices are labeled by the binary n -tuples $(a_1, a_2, \dots, a_n)$ [that is $a_i = 0$ or 1 for $1 \leq i \leq n$] such that two vertices are adjacent if and only if their corresponding n -tuples differ in exactly one position. Note that the

hypercube Q_n is a bipartite graph. To every vertex $v = (a_1, a_2, \dots, a_n) \in Q_n$, we assign a number $n(v) = a_1 + a_2 + \dots + a_n$. Then the two partite sets are $S_n = \{v \in Q_n : n(v) \equiv 0 \pmod{2}\}$ and $T_n = \{v \in Q_n : n(v) \equiv 1 \pmod{2}\}$. Clearly, the graph Q_n has 2^n vertices and $n2^{n-1}$ edges.

Definition 2.3. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M .

Theorem 2.4. *Let M be any perfect matching of a graph G . Then in any friendly coloring of G an even number of elements of M are labeled 0.*

Proof. We note that every graph with perfect matching has even number of vertices. Let $M = \{e_1, e_2, \dots, e_n\}$ with $e_i = u_i v_i$ be a perfect matching of G and $f : V(G) \rightarrow \mathbb{Z}_2$ an arbitrary friendly coloring of G . If $f_*(e_i) = 1$ for all $i = 1, 2, \dots, n$, then the statement of the theorem is true. Assume, without loss of generality, that $f_*(e_1) = f_*(e_2) = \dots = f_*(e_k) = 0$ ($1 \leq k \leq n$). Then $f_*(e_{k+1}) = \dots = f_*(e_n) = 1$, which implies that $n - k$ vertices of G are labeled 0 and another $n - k$ vertices are labeled 1. For f to be a friendly coloring, k has to be even, say $k = 2r$, so that r elements of $M = \{e_1, e_2, \dots, e_k\}$ be adjacent to vertices labeled 1 and another r elements be adjacent to vertices labeled 0. $\square$

Lemma 2.5. *The hypercube Q_n has n disjoint perfect matchings, each one has 2^{n-1} elements.*

Proof. we proceed by induction. The statement is true for $n = 1, 2, 3$. Suppose it is true for Q_{n-1} and let M be a perfect matching of Q_{n-1} . Then $M \cup M'$ is a perfect matching of Q_n , where M' is the corresponding perfect matching of the second copy of Q_{n-1} . This provides $n - 1$ perfect matching for Q_n . The other one is the natural matching $\{v_i w_i : 1 \leq i \leq 2^{n-1}\}$ as outlined in Remark 2.1. $\square$

Corollary 2.6. *For any binary friendly coloring of Q_n ($n \geq 2$), $e_f(0)$ and $e_f(1)$ are even.*

Corollary 2.7. *For $n \geq 2$ every element of $C(Q_n)$ is a multiple of 4.*

Proof. As a result of 2.4 and 2.5, the induced edge labeling of any friendly coloring $f : V(Q_n) \rightarrow \mathbb{Z}_2$ labels an even number of edges of Q_n by 0, that is, $e_f(0)$ is even. Therefore, by (1.1), the number $N(f) = n2^{n-1} - 2e_f(0)$ is a multiple of 4. $\square$

Theorem 2.8. *For any friendly coloring $f : V(Q_n) \rightarrow \mathbb{Z}_2$ that is not maximum $N(f) \neq n2^{n-1}$,*

$$N(f) \leq n2^{n-1} - 4(n - 1).$$

Proof. We observe that there are six different ways of coloring the vertices of an square Q_2 by elements of $\mathbb{Z}_2$, as illustrated in Figure 3. In each of these colorings, which are not necessarily friendly, the number of edges labeled 1 (or 0) is even.

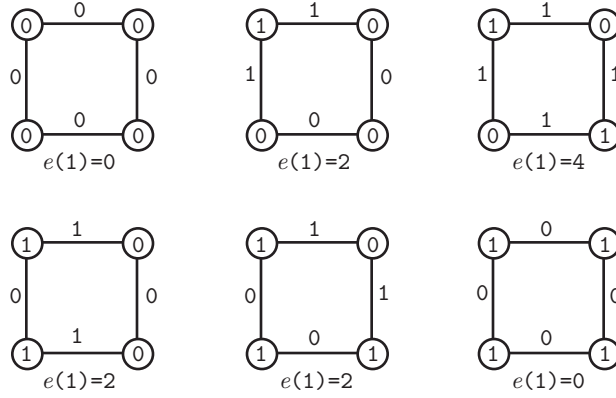


Figure 3: Six possible ways of binary colorings of Q_2 .

Now for any friendly coloring $f : V(Q_n) \rightarrow \mathbb{Z}_2$, its induced edge labeling will label at least one edge by 1. Since $e_f(1)$ is even, we have $e_f(1) \geq 2$. Therefore, there are at least two edges labeled 1. We can assume that these edges are disjoint. These two edges are shared by $n - 1$ squares and even number of edges of these squares are labeled 1, which implies that $e_f(1) \geq 2(n - 1)$. Therefore, by (1.1), $N(f) = n2^{n-1} - 2e_f(1) \leq n2^{n-1} - 4(n - 1)$. As we shall see later, in Lemma 3.1 part (b), there is a friendly coloring that produces the index $n2^{n-1} - 4(n - 1)$. $\square$

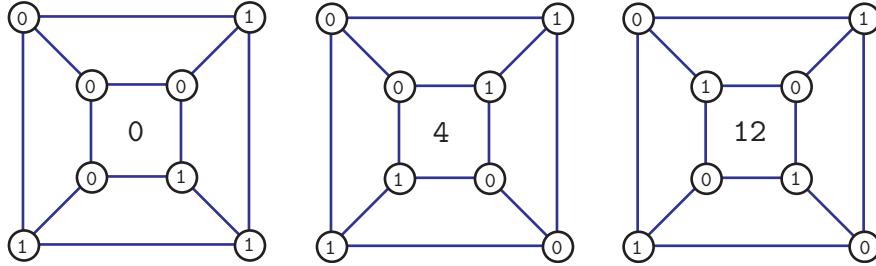


Figure 4: $C(Q_3) = \{0, 4, 12\}$.

Since the cordial set of Q_1 is $\{1\}$, $C(Q_2) = \{0, 4\}$, and $C(Q_3) = \{0, 4, 12\}$, from this point on we assume that $n \geq 4$. For the proof the following two theorems, we use the notations of the Remark 2.1.

Theorem 2.9. *If $x \in C(Q_{n-1})$, then $2x + 2^{n-1} \in C(Q_n)$.*

Proof. Let $x = N(f)$, where f is a friendly coloring of Q_{n-1} . We consider two cases:

- (1) $N(f) = e_f(0) - e_f(1)$. Let $g : V(Q_n) \rightarrow \mathbb{Z}_2$ be defined by $g(u_i, a) = g(u_i, b) = f(u_i)$ for all $u_i \in V(Q_{n-1})$. Then g is a friendly coloring of Q_n with $N(g) = 2N(f) + |V(Q_{n-1})| = 2x + 2^{n-1}$.

- (2) $N(f) = e_f(1) - e_f(0)$. Let $h : V(Q_n) \rightarrow \mathbb{Z}_2$ be defined by $h(u_i, a) = f(u_i)$ and $h(u_i, b) = 1 - f(u_i)$ for all $u_i \in V(Q_{n-1})$. Then h is a friendly coloring of Q_n with $N(h) = 2N(f) + |V(Q_{n-1})| = 2x + 2^{n-1}$.

□

Theorem 2.10. *If $x \in C(Q_{n-1})$, then $|2x - 2^{n-1}| \in C(Q_n)$.*

Proof. Let $x = N(f)$, where f is a friendly coloring of Q_{n-1} . We consider two cases:

- (1) $N(f) = e_f(1) - e_f(0)$. Let $g : V(Q_n) \rightarrow \mathbb{Z}_2$ be defined by $g(u_i, a) = g(u_i, b) = f(u_i)$ for all $u_i \in V(Q_{n-1})$. Then g is a friendly coloring of Q_n with $e_g(1) = 2e_f(1)$ and $e_g(0) = 2e_f(0) + 2^{n-1}$. Therefore, $N(g) = |e_g(1) - e_g(0)| = |2e_f(1) - 2e_f(0) - 2^{n-1}| = |2x - 2^{n-1}|$.
- (2) $N(f) = e_f(0) - e_f(1)$. Let $h : V(Q_n) \rightarrow \mathbb{Z}_2$ be defined by $h(u_i, a) = f(u_i)$ and $h(u_i, b) = 1 - f(u_i)$ for all $u_i \in V(Q_{n-1})$. Then h is a friendly coloring of Q_n with $N(h) = |2N(f) - 2^{n-1}| = |2x - 2^{n-1}|$.

□

Theorem 2.11. *For every $n > 1$, $0 \in C(Q_n)$; that is, hypercubes are cordial*

Proof. We proceed by strong induction on n . The statement is true for $n = 2, 3$. Suppose it is true for all k with $2 \leq k < n$. We wish to show that $0 \in C(Q_n)$. Note that since $0 \in C(Q_{n-2})$, by Theorem 2.10 with $x = 0$, $2^{n-2} \in C(Q_{n-1})$. Therefore, by Theorem 2.10 with $x = 2^{n-2}$, $0 = 2x - 2^{n-1} \in C(Q_n)$.

□

3 Cordial Sets of Hypercubes

In this section we further investigate about the elements of cordial sets of hypercubes and shall completely determine $C(Q_4)$.

Lemma 3.1. *For any $n \geq 3$,*

- (a) $n2^{n-1} \in C(Q_n)$. *This is the maximum cordial index of Q_n .*
- (b) $n2^{n-1} - 4(n-1) \in C(Q_n)$.
- (c) $n2^{n-1} - 4n \in C(Q_n)$.
- (d) $|n2^{n-1} - 8(n-1)| \in C(Q_n)$.

(e) $|n2^{n-1} - 8(n-2)| \in C(Q_n)$.

Proof. For proof of this lemma we utilize the notations mentioned in Remark 2.2 and assume that $u_1 = (0, 0, \dots, 0)$, $u_2 = (0, 0, 1, 0, \dots, 0)$, $u_3 = (1, 0, 1, 0, \dots, 0)$, $u_4 = (1, 0, \dots, 0)$, $u_5 = (1, 1, 0, \dots, 0)$ and $u_6 = (1, 1, 1, 0, \dots, 0)$.

(a) Define $f : V(Q_n) \rightarrow \mathbb{Z}_2$ by $f(x) = \begin{cases} 1 & \text{if } x \in S_n; \\ 0 & \text{if } x \in T_n. \end{cases}$ Then f is a friendly coloring with $e_f(1) = n2^{n-1}$, $e_f(0) = 0$. Therefore, $N(f) = n2^{n-1}$.

(b) Consider the two adjacent vertices u_1, u_2 and define $\alpha : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\alpha(x) = \begin{cases} 1 - f(x) & \text{if } x \in \{u_1, u_2\}; \\ f(x) & \text{otherwise,} \end{cases}$$

where f is the function defined in part (a). Then α is a friendly coloring with $e_\alpha(0) = 2(n-1)$. Therefore, by (1.1), $N(\alpha) = n2^{n-1} - 2e_\alpha(0) = n2^{n-1} - 4(n-1)$.

(c) Consider the two nonadjacent vertices u_1, u_6 and define $\beta : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\beta(x) = \begin{cases} 1 - f(x) & \text{if } x \in \{u_1, u_6\}; \\ f(x) & \text{otherwise,} \end{cases}$$

where f is the function defined in part (a). Then β is a friendly coloring with $e_\beta(0) = 2n$. Therefore, by (1.1), $N(\beta) = n2^{n-1} - 2e_\beta(0) = n2^{n-1} - 4n$.

(d) Consider four vertices $A = \{u_1, u_2, u_5, u_6\}$ and define $\gamma : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\gamma(x) = \begin{cases} 1 - f(x) & \text{if } x \in A; \\ f(x) & \text{otherwise,} \end{cases}$$

where f is the function defined in part (a). Then γ is a friendly coloring with $e_\gamma(0) = 4(n-1)$. Therefore, by (1.1), $N(\gamma) = |n2^{n-1} - 2e_\gamma(0)| = |n2^{n-1} - 8(n-1)|$.

(e) Consider four vertices $B = \{u_1, u_2, u_3, u_4\}$ and define $\lambda : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\lambda(x) = \begin{cases} 1 - f(x) & \text{if } x \in B; \\ f(x) & \text{otherwise,} \end{cases}$$

where f is the function defined in part (a). Then λ is a friendly coloring with $e_\lambda(0) = 4(n-2)$. Therefore, by (1.1), $N(\lambda) = |n2^{n-1} - 2e_\lambda(0)| = |n2^{n-1} - 8(n-2)|$.

□

Lemma 3.2. For any $n \geq 3$,

- (a) $(n-2)2^{n-1} \in C(Q_n)$.
- (b) $(n-2)2^{n-1} - 4(n-1) \in C(Q_n)$.
- (c) $(n-2)2^{n-1} - 4(n-2) \in C(Q_n)$.
- (d) $|(n-2)2^{n-1} - 8(n-1)| \in C(Q_n)$.
- (e) $|(n-2)2^{n-1} - 8(n-2)| \in C(Q_n)$.

Proof. For proof of this lemma we utilize the notations mentioned in Remark 2.1.

- (a) Define $g : V(Q_n) \rightarrow \mathbb{Z}_2$ by $g(v) = \begin{cases} 1 & \text{if } v \in H; \\ 0 & \text{if } v \in K. \end{cases}$ Then g is a friendly coloring with $e_g(1) = |V(Q_{n-1})| = 2^{n-1}$. Therefore, by (1.1), $N(g) = n2^{n-1} - 2e_g(1) = (n-2)2^{n-1}$.
- (b) Consider the edge v_1w_1 and define $\alpha : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\alpha(x) = \begin{cases} 1 - g(x) & \text{if } x \in \{v_1, w_1\}; \\ g(x) & \text{otherwise,} \end{cases}$$

where g is the function defined in part (a). Then α is a friendly coloring with $e_\alpha(1) = |V(Q_{n-1})| - 2(n-1)$. Therefore, by (1.1), $N(\alpha) = n2^{n-1} - 2e_\alpha(1) = (n-2)2^{n-1} - 4(n-1)$.

- (c) Let F be the subgraph of Q_n induced by vertices v_1, v_2, w_1, w_2 , where $v_1v_2, w_1w_2 \in E(Q_n)$. Define $\beta : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\beta(x) = \begin{cases} 1 - g(x) & \text{if } x \in \{v_1, w_2\}; \\ g(x) & \text{otherwise,} \end{cases}$$

where g is the function defined in part (a). Then β is a friendly coloring with

$$e_\beta(1) = |V(Q_{n-1})| - 2 + 2(n-1). \text{ Therefore, by (1.1),}$$

$$N(\beta) = n2^{n-1} - 2e_\beta(1) = (n-2)2^{n-1} - 4(n-2).$$

- (d) For this part, we choose four vertices $A = \{v_1, v_2, w_1, w_2\}$ such that $d(v_1, v_2) \geq 3$, and define $\gamma : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\gamma(x) = \begin{cases} 1 - g(x) & \text{if } x \in A; \\ g(x) & \text{otherwise,} \end{cases}$$

where g is the function defined in part (a). Then γ is a friendly coloring with

$$e_\gamma(1) = |V(Q_{n-1})| + 4(n-1). \text{ Therefore, by (1.1),}$$

$$N(\gamma) = |n2^{n-1} - 2e_\gamma(1)| = |(n-2)2^{n-1} - 8(n-1)|.$$

(e) With notation of part (c) define $\lambda : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\lambda(x) = \begin{cases} 1 - g(x) & \text{if } x \in V(F); \\ g(x) & \text{otherwise,} \end{cases}$$

where g is the function defined in part (a). Then λ is a friendly coloring with

$$e_\lambda(1) = |V(Q_{n-1})| + 4(n-2). \text{ Therefore, by (1.1),}$$

$$N(\lambda) = |n2^{n-1} - 2e_\lambda(1)| = |(n-2)2^{n-1} - 8(n-2)|.$$

□

Observation 3.3. For $n \geq 3$ and $1 \leq k \leq 2^{n-2}$, $(n-2)|2^{n-1} - 4k| \in C(Q_n)$.

Proof. Using the notations of 2.2, let S_{n-1} and T_{n-1} be the partite sets of Q_{n-1} . Also, let $X = \{x_1, x_2, \dots, x_k\} \subseteq S_{n-1}$, $Y = \{y_1, y_2, \dots, y_k\} \subseteq T_{n-1}$ and $K_2 = \{a, b\}$. Define $\varphi : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\varphi(u) = \begin{cases} 0 & \text{if } u = (u_i, a) \text{ and } u_i \notin X; \\ 0 & \text{if } u = (u_i, b) \text{ and } u_i \in Y; \\ 1 & \text{otherwise.} \end{cases}$$

Then φ is a friendly coloring with $e_\varphi(1) = |V(Q_{n-1})| - 2k + 2k(n-1) = (n-1)2^{n-2} + 2k(n-2)$.

Therefore, $N(\varphi) = |n2^{n-1} - 2e_\varphi(1)| = (n-2)|2^{n-1} - 4k|$. □

At this juncture, we utilize the facts developed in this and previous sections to determine the cordial set of Q_4 .

To begin with, we note that by 2.7 all members of $C(Q_4)$ are multiple of 4. Next, since $0, 4, 12 \in C(Q_3)$, then by 2.9 and 2.10, $0, 8, 16, 32 \in C(Q_4)$. Also, by 3.1(c), $20 \in C(Q_4)$. In addition, by 3.2(b), $4 \in C(Q_4)$. It only remains to show that $12 \in C(Q_4)$ and this is illustrated in Figure 3. Finally, by 2.8, we know that $24, 28 \notin C(Q_4)$. Therefore, $C(Q_4) = \{0, 4, 8, 12, 16, 20, 32\}$.

4 Conjecture

Conjecture 4.1 (Salehi). For $n \geq 3$,

$$C(Q_n) = \{4k : 0 \leq k \leq n2^{n-3} - n + 1\} \cup \{n2^{n-1}\}.$$

The validity of this conjecture was already verified for $n = 2, 3, 4$. Moreover, it is also verified for $n = 5$ by a computer program. This verification is summarized in the following table where the first column lists the 32 vertices of Q_5 in binary notation. The other columns provide the friendly coloring of Q_5 with the cordial indices that are given as the entries of the last row.

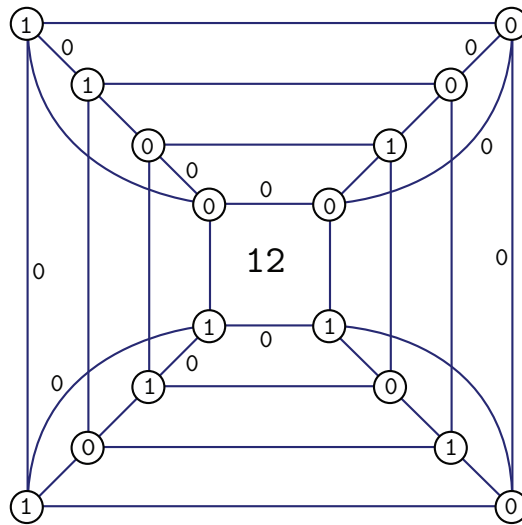


Figure 5: A friendly labeling of Q_4 with $e_f(0) = 10$ which provides index 12.

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Vertices	18 different colorings of 5-cube																	
00000	0	0	0	0	0	0	1	1	0	0	1	1	1	1	1	0	1	1
00001	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0	1	0	0
00010	1	0	1	1	0	1	0	0	0	1	0	0	0	0	0	1	0	0
00011	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1	0	1	1
00100	0	1	1	0	0	1	1	0	0	1	0	1	0	0	0	1	0	0
00101	1	1	1	0	1	1	0	0	0	0	1	1	1	1	1	0	1	1
00110	1	0	1	1	1	1	1	1	0	1	1	0	1	1	1	0	1	1
00111	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
01000	0	0	0	0	0	0	1	1	1	0	0	0	1	0	0	1	0	0
01001	0	1	1	0	1	1	0	0	1	0	1	1	1	1	1	0	1	1
01010	0	1	0	0	0	1	0	0	0	0	1	1	1	1	1	0	1	1
01011	1	1	1	0	0	0	1	1	0	1	0	0	0	0	0	1	0	0
01100	1	1	1	1	1	0	0	1	0	0	1	0	1	1	1	0	1	1
01101	0	1	1	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0
01110	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0
01111	0	0	1	1	1	1	0	1	0	0	1	1	1	1	1	0	1	1
10000	1	0	1	0	1	1	0	0	1	1	1	0	0	0	0	1	0	0
10001	0	1	1	1	0	0	1	1	1	0	0	1	0	1	1	0	1	1
10010	1	0	1	0	1	0	0	1	1	0	1	1	1	1	1	1	1	1
10011	1	0	0	0	1	1	1	0	1	0	0	0	0	0	0	1	0	0
10100	0	0	0	1	0	1	1	1	0	0	1	1	1	1	1	0	1	1
10101	1	1	0	0	1	0	0	0	0	1	1	0	1	0	0	0	0	0
10110	0	1	0	1	0	0	1	0	1	1	0	1	0	0	0	1	0	0
10111	1	1	0	0	0	1	0	1	1	0	1	1	1	1	1	0	1	1
11000	0	0	0	1	1	0	0	1	1	0	1	1	0	0	1	0	1	1
11001	1	0	0	0	0	1	0	1	1	1	1	0	0	0	0	1	0	0
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11101	1	1	0	1	1	0	1	1	1	0	0	1	1	0	0	0	1	1
11110	0	1	0	1	0	1	0	0	1	0	1	0	1	1	0	0	0	1
11111	1	0	0	1	1	0	1	0	1	1	0	0	0	1	1	1	1	0
Indices	0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	80